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1987 J. Phys. A: Math. Gen. 20 L653

(http://iopscience.iop.org/0305-4470/20/10/006)

J. Phys. A: Math. Gen. 20 (1987) L653–L657. Printed in the UK

LETTER TO THE EDITOR

Finite-size corrections to matrix elements in a conformal theory. Applications to the magnetisation of the three-state Potts model

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Received 20 March 1987

Abstract. Using conformal invariance we calculate the leading corrections in $1/N$ to the magnetisation of the three-state Potts quantum chain. The results are in good agreement with our numerical finite-size results.

In a recent paper, one of us has presented a method of calculating corrections to the conformal spectrum of quantum chains with a finite number of sites (Reinicke 1987). Since matrix elements are a better test than energy eigenvalues, we present in this letter the leading corrections to the magnetisation of the three-state Potts model obtaining good agreement with numerical results.

The Hamiltonian of the three-state Potts quantum chain is given by

$$H = -\frac{2}{3\sqrt{3}} \sum_{i=1}^{N} [\sigma_i + \sigma_i^+ + \lambda\{\Gamma_i \Gamma_{i+1}^+ + \Gamma_i^+ \Gamma_{i+1}\}] \tag{1}$$

where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \omega = \exp(\tfrac{2}{3}\pi i). \tag{2}$$

Here λ has the meaning of the inverse temperature and N is the number of sites. We consider the Hamiltonian at its critical point, $\lambda = 1$, and choose periodic boundary conditions $\Gamma_{N+1} = \Gamma_1$. Since the Hamiltonian (1) is $S_3 = Z_3 \otimes_s Z_2$ symmetric it has block diagonal form H_Q($Q = 0, 1, 2$ denoting the charge sector), with $H_1 = H_2$. Let $|i\rangle$ denote the lowest state of each sector. Hamer (1982) has shown that the magnetisation is related to the greatest eigenvalue λ_N of the matrix

$$\langle i|\tfrac{1}{2}(\Gamma_l + \Gamma_l^+)|j\rangle. \tag{3}$$

Notice that these elements are independent of the site l, and that the diagonal ones vanish. Let

$$\begin{aligned} a_N &= \langle 0|\Gamma_l + \Gamma_l^+|1\rangle = \langle 0|\Gamma_l + \Gamma_l^+|2\rangle \\ b_N &= \langle 1|\Gamma_l + \Gamma_l^+|2\rangle. \end{aligned} \tag{4}$$

Then a_N, b_N and λ_N are related by

$$\lambda_N = \frac{b_N}{4}\left\{1 + \left[1 + 8\left(\frac{a_N}{b_N}\right)^2\right]^{1/2}\right\}. \tag{5}$$

0305-4470/87/100653+05\$02.50

Hamer (1982) gave the values of λ_N for $N \leq 10$ and found by means of a Van den Broeck and Schwartz (1979) extrapolation $\lambda_N \propto N^{-2/15}$ with an accuracy of 10^{-5} for the exponent. Table 1 shows our values for a_N and b_N for $N \leq 8$. These quantities also show an $N^{-2/15}$ behaviour although the accuracy for the exponent is only 10^{-3} (it seems that for λ_N some corrections are cancelled).

In order to obtain the next correction we set

$$\begin{aligned} \lambda_N &= N^{-2/15}(A + BN^{-\omega} + \ldots) \\ a_N &= N^{-2/15}(A_1 + B_1 N^{-\omega} + \ldots) \\ b_N &= N^{-2/15}(A_2 + B_2 N^{-\omega} + \ldots) \end{aligned} \tag{6}$$

where in view of (5) we choose the same exponent ω. Using the extrapolations of Van den Broeck and Schwartz (1979) for the λ_N we obtain (the convergence is here much better)

$$A = 1.032\,515\,(5) \qquad B = 0.029\,10\,(5) \qquad \omega = 0.8000\,(5). \tag{7}$$

Now using $\omega = 0.8$ one obtains from the values of a_N and b_N

$$\begin{aligned} A_1 &= 1.001\,42\,(7) \qquad & B_1 &= +0.0083\,(5) \\ A_2 &= 1.093\,90\,(6) \qquad & B_2 &= -0.1027\,(5). \end{aligned} \tag{8}$$

Notice that the values of A and B are consistent with (5).

Now we calculate the ratio B_1/A_1 using conformal invariance for the infinite chain (the calculation of the ratio B_2/A_2 requires a four-point function, which is not known). Let us remind the reader that the spectrum of the three-state Potts model at $\lambda = 1$ and $N = \infty$ can be described by the irreducible representations (IR) of two commuting Virasoro algebras with central charge $c = \frac{4}{5}$ (Friedan *et al* 1984, Dotsenko 1984). We denote by Δ the highest weight, and by $\Delta + r$ the rth level having a degeneracy $d(\Delta, r)$ of one IR of the Virasoro algebra. A state will be labelled by $|\Delta + r, \bar{\Delta} + \bar{r}; i\rangle$; $i = 1, \ldots, d(\Delta, r)d(\bar{\Delta}, r)$. For periodic boundary conditions the scaled spectra $\mathscr{E}_Q(Q = 0,$ 1, 2 denotes the sector) can be described by the following sums of the IR of the Virasoro algebras (von Gehlen and Rittenberg 1986, Cardy 1986a):

$$\begin{aligned} &\mathscr{E}_0 = (0,0) \oplus (\tfrac{2}{5}, \tfrac{2}{5}) \oplus (\tfrac{2}{5}, \tfrac{7}{5}) \oplus (\tfrac{7}{5}, \tfrac{2}{5}) \oplus (3,0) \oplus (0,3) \oplus (3,3) \\ &\mathscr{E}_1 = (\tfrac{1}{15}, \tfrac{1}{15}) \oplus (\tfrac{2}{3}, \tfrac{2}{3}) \qquad \mathscr{E}_2 = (\tfrac{1}{15}^+, \tfrac{1}{15}^+) \oplus (\tfrac{2}{3}^+, \tfrac{2}{3}^+). \end{aligned} \tag{9}$$

Let us denote by σ and σ^+ the primary fields with scaling dimensions $\Delta = \bar{\Delta} = \frac{1}{15}$, where σ (and its conformal family) has charge one, σ^+ charge two. The primary fields ρ and

Table 1. The matrix elements a_N and b_N defined in (4) for various number of sites N.

N	a_N	b_N
2	0.915 6158	0.945 3795
3	0.867 4123	0.908 6855
4	0.834 4876	0.881 4460
5	0.809 7724	0.859 9376
6	0.790 1242	0.842 2460
7	0.773 8928	0.827 2674
8	0.760 1111	0.814 3100

ρ^+, both with dimensions $\Delta=\bar{\Delta}=\frac{2}{3}$ have charge one and two respectively. Since the matrix $\Gamma(\Gamma^+)$ takes a state with charge j to a state with charge $j-1$ $(j+1)$, it has to be a linear combination of σ^+, $\rho^+(\sigma,\rho)$ and their conformal families, so that

$$\begin{aligned}\Gamma&=a_0\sigma^+(0,0)+a_1[(L_{-1}\sigma^+)(0,0)+(\bar{L}_{-1}\sigma^+)(0,0)]+\ldots+b_0\rho^+(0,0)+\ldots\\ \Gamma^+&=a_0\sigma(0,0)+a_1[(L_{-1}\sigma)(0,0)+(\bar{L}_{-1}\sigma)(0,0)]+\ldots+b_0\rho(0,0)+\ldots\end{aligned}\tag{10}$$

where, due to charge conjugation, we have the same coefficients in the expansion of Γ and Γ^+. We consider the fields on the strip. Since the lowest state in the charge sector one $|1\rangle^c$ corresponds to the state $|\frac{1}{15},\frac{1}{15}\rangle$ of the conformal theory, we have for N going to infinity ($|1\rangle\to|1\rangle^c$) (Cardy 1986b)

$$\begin{aligned}{}^c\langle 0|\sigma(0,0)+\sigma^+(0,0)|1\rangle^c&={}^c\langle 0|\sigma^+(0,0)|1\rangle^c=\left(\frac{2\pi}{N}\right)^{2/15}={}^c\langle 1|\sigma(0,0)|0\rangle^c\\ \langle 2|\sigma(0,0)+\sigma^+(0,0)|1\rangle^c&=\left(\frac{2\pi}{N}\right)^{2/15}(C_{1/15,1/15,1/15})^2\end{aligned}\tag{11}$$

where $C^{\Delta_3^+}_{\Delta_1,\Delta_2}\equiv C_{\Delta_1,\Delta_2,\Delta_3}$ is the coefficient appearing in the short distance expansion

$$\varphi_{\Delta_1}(Z_1)\varphi_{\Delta_2}(Z_2)=\sum_{\Delta_3^+}C^{\Delta_3^+}_{\Delta_1,\Delta_2}(Z_1-Z_2)^{\Delta_3^+-\Delta_1-\Delta_2}\varphi_{\Delta_3^+}(Z_2)+\ldots.$$

Notice that taking secondary fields of the conformal families of σ and σ^+ one obtains an $N^{-2/15-k}$ dependence, where k is a positive integer. (From the families of ρ and ρ^+ one obtains an $N^{-4/3-k}$ dependence.) From equations (4), (6), (8) and (11) we have

$$a_0(2\pi)^{2/15}=A_1=1.001\,42\,(7)\qquad (C_{1/15,1/15,1/15})^2=A_2/A_1=1.0924\,(1).\tag{12}$$

For a finite number of sites N, we expect corrections to the states. In general the Hamiltonian will be changed by irrelevant operators (Cardy 1986b)

$$H=H^c+\sum_i\gamma_i\int_{-N/2}^{N/2}\mathrm{d}v\,\varphi_i(0,v)\tag{13}$$

where γ_i are parameters and φ_i are local fields of the conformal theory having scaling dimensions $(\Delta_i+r_i,\bar{\Delta}_i+\bar{r}_i)$, with $g_i=\Delta_i+r_i+\bar{\Delta}_i+\bar{r}_i-2>0$ and $g_{i+1}\geqslant g_i$. In first-order perturbation theory one obtains from (13) corrections to a state proportional to N^{-g_i} (Reinicke 1987). In order to simplify the formulae we assume that $g_0<g_1$ and that φ_0 is a primary field with dimensions $\Delta_0=\bar{\Delta}_0=:\Delta$. Suppose we want to obtain the leading corrections to the matrix element $\langle 0|\psi^+(0,0)|\Delta',\Delta'\rangle$, where ψ^+ is a primary field with scaling dimensions $\Delta'=\bar{\Delta}'$. By means of the method exposed in a previous paper (Reinicke 1987) we have

$$\begin{aligned}\langle 0|\psi^+(0,0)|\Delta',\Delta'\rangle&={}^c\langle 0|\psi^+(0,0)|\Delta',\Delta'\rangle^c\\ &\quad-\gamma_0\sum_{i\neq 0}(E_i^c-E_0^c)^{-1}\,{}^c\left\langle 0\left|\int_{-N/2}^{N/2}\mathrm{d}v\,\varphi_0(0,v)\right|i\right\rangle^c\,{}^c\langle i|\psi^0(0,0)|\Delta',\Delta'\rangle^c\\ &\quad-\gamma_0\sum_{i\neq(\Delta',\Delta')}(E_i^c-E^c_{\Delta',\Delta'}|^{-1}\,{}^c\langle 0|\psi^+(0,0)|i\rangle^c\,{}^c\left\langle i\left|\int_{-N/2}^{N/2}\mathrm{d}v\,\varphi_0(0,v)\right|\Delta',\Delta'\right\rangle^c\\ &={}^c\langle 0|\psi^+(0,0)|\Delta',\Delta'\rangle^c\left[1-\frac{\gamma_0}{2}(C_{\Delta',\Delta'^+,\Delta})^2(2\pi)^{2\Delta-1}N^{2-2\Delta}\right.\\ &\quad\left.\times\left(\int_0^1\mathrm{d}x\,x^{\Delta-1}F(\Delta,\Delta;1;x)+\int_0^1\mathrm{d}x\frac{1}{x}(F(\Delta,\Delta;1;x)-1)\right)\right]\end{aligned}\tag{14}$$

where F is the hypergeometric function. In detail

$$\sum_{i\neq 0}(E_i^c-E_0^c)^{-1}\,{}^c\left\langle 0\left|\int_{-N/2}^{N/2}\mathrm{d}v\,\varphi_0(0,v)\right|i\right\rangle^c\,{}^c\langle i|\psi^+(0,0)|\Delta',\Delta'\rangle^c$$

$$=\int_0^\infty \mathrm{d}\tau\int_{-N/2}^{N/2}\mathrm{d}v\sum_{i\neq 0}\exp[-(E_i^c-E_0^c)\tau]\,{}^c\langle 0|\varphi_0(0,v)|i\rangle^c\,{}^c\langle i|\psi^+(0,0)|\Delta',\Delta'\rangle^c$$

$$=\int_0^\infty \mathrm{d}\tau\int_{-N/2}^{N/2}\mathrm{d}v\lim_{\tau_3\to 0}\exp[(E^c_{\Delta',\Delta'}-E_0^c)\tau_3]$$

$$\times\frac{{}^c\langle 0|\varphi_0(-\tau,v)\psi^+(0,0)\psi(\tau_3,v_3)|0\rangle^c}{{}^c\langle\Delta',\Delta'|\psi(0,0)|0\rangle^c}$$

$$=(C_{\Delta',\Delta'^+,\Delta})^2\left(\frac{2\pi}{N}\right)^{2\Delta}{}^c\langle 0|\psi^+(0,0)|\Delta',\Delta'\rangle^c\int_0^\infty\mathrm{d}\tau\int_{-N/2}^{N/2}\mathrm{d}v$$

$$\times\frac{\exp(-2\pi\tau\Delta/N)}{[2\cosh(2\pi\tau/N)-2\cos(2\pi v/N)]^\Delta}$$

$$=\tfrac{1}{2}(C_{\Delta',\Delta'^+,\Delta})^2(2\pi)^{2\Delta-1}N^{2-2\Delta}\int_0^1\mathrm{d}x\,x^{\Delta-1}F(\Delta,\Delta;1;x). \qquad (15)$$

The integrals in (14) are divergent for $\Delta>1$, but can easily be regularised. For $0<\Delta<1$ we have after $2K$ partial integrations

$$\int_0^1\mathrm{d}x\,x^{\Delta-1}F(\Delta,\Delta;1;x)+\int_0^1\mathrm{d}x\,x^{-1}[F(\Delta,\Delta;1;x)-1]$$

$$=\sum_{\nu=1}^{K}\left[\frac{\Gamma(2\nu-2\Delta)}{\Gamma^2(\nu+1-\Delta)}\left(\frac{(\Delta-3\nu)\Gamma^2(\nu)\Gamma^2(1-\Delta)}{\Gamma(\nu-\Delta)\Gamma(1+\nu-\Delta)}-3\right)-\frac{2}{\Delta-\nu}\right]$$

$$+\frac{(K!)^2\Gamma^2(1-\Delta)}{\Gamma^2(K+1-\Delta)}\int_0^1\mathrm{d}x\,x^{\Delta-1}F(\Delta-K,\Delta-K;1;x)$$

$$+\int_0^1\frac{\mathrm{d}x}{x}[F(\Delta-K,\Delta-K;1;x)-1]. \qquad (16)$$

The right-hand side of this equation is perfectly well defined for $K<\Delta<K+1$ (the limit $\Delta\to K+\frac{1}{2}$ is finite).

Now we return to the magnetisation of the three-state Potts model. It was established numerically (von Gehlen *et al* 1987) that φ_0 is the primary field with $\Delta=\bar{\Delta}=\frac{7}{5}$, and $\gamma_0=0.009\,237\,(7)$ $(g_1=2>g_0=\frac{4}{5})$. The constant $(C_{1/15^+,1/15,7/5})^2$ was also determined. Inserting these values into (14) and using (16), one has

$$\langle 0|\sigma^+(0,0)|1\rangle=\left(\frac{2\pi}{N}\right)^{2/15}\left(1+\frac{0.006\,739\,(5)}{N^{0.8}}+\ldots\right) \qquad (17)$$

where the numerical error is due to the error in γ_0. This reproduces the value of $B_1/A_1=0.0083\,(5)$ of (8) within 20%. The ratio B_2/A_2 could be determined in principle using the same method. However, in this case one needs the four-point function $\langle\sigma^+\varphi_0\sigma^+\sigma^+\rangle$—which is not known—instead of the three-point function $\langle\varphi_0\sigma^+\sigma\rangle$.

To conclude we want to give a list of the coefficients $C_{\Delta_1,\Delta_2,\Delta_3}$ (partially published by von Gehlen *et al* (1987)) for the three-state Potts model. Apart from the trivial coefficients $C_{\Delta,\Delta^+,0}=1$ there are essentially 12 non-vanishing coefficients, namely

$$\begin{gathered} C_{7/5,2/5,2/5}=-\tfrac{2}{3}C_{7/5,7/5,7/5}=-\tfrac{1}{6}C_{7/5,1/15,1/15^+} \\ 2(C_{2/5,1/15,1/15^+})^2=\tfrac{7}{6}(C_{7/5,2/5,2/5})^2=\left(\frac{\Gamma(3/5)}{\Gamma(2/5)}\right)^{3/2}\left(\frac{\Gamma(1/5)}{\Gamma(4/5)}\right)^{1/2}=:X=1.092\,436\ldots \\ (C_{2/3,2/3^+,3})^2=-\tfrac{13}{7}(\tfrac{2}{3})^4 \qquad (C_{7/5,2/5,3})^2=\tfrac{21}{26} \qquad (C_{2/5,1/15,2/3^+})=\tfrac{2}{3} \\ C_{1/15,1/15,1/15} \qquad C_{2/3,2/3,2/3} \qquad C_{1/15,1/15,2/3} \qquad C_{7/5,1/15,2/3^+} \qquad C_{1/15,1/15^+,3} \end{gathered} \tag{18}$$

where the last ones are not known. (Notice that the coefficients are symmetric under any interchange of the Δ and that $C_{\Delta_1,\Delta_2,\Delta_3}=C_{\Delta_1^+,\Delta_2^+,\Delta_3^+}$.) Considering the operator content of cyclic boundary conditions together with periodic ones (von Gehlen and Rittenberg 1986) one can recognise a multiplet structure given by two multiplets $(0,3,\frac{2}{3},\frac{2}{3}^+)$ and $(\frac{7}{5},\frac{2}{5},\frac{1}{15},\frac{1}{15}^+)$. From (18) we see that for Δ_i $(i=1,2,3)$ belonging to the second multiplet $(C_{\Delta_1,\Delta_2,\Delta_3})^2$ always has the structure 'simple rational number times X'. From this and (12) we are tempted to set $(C_{1/15,1/15,1/15})^2=X$, where X is given in (18).

We are grateful to V Rittenberg for stimulating discussions and substantial support.

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